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A Continuous Analog of Ford–Fulkerson Flows in Networks and Its Application to a Problem of Rota

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1. INTRODUCTION

Given a poset Q, a natural question to ask is "What is the largest subset of Q having no comparable elements?" In 1928 E. Sperner [SP] showed that for $Q = B_n$, the set of all subsets of an n-set ordered by containment, the answer is $B_{n,\lfloor n/2 \rfloor}$, its largest rank. Subsequently, sets of incomparable elements have become known as Sperner sets; the general maximization problem as the Sperner problem, and the property of a graded poset that the solution be the largest rank as the Sperner property.

In 1970 G.-C. Rota [RO] asked if Π_n, the lattice of partitions of an n-set, ordered by refinement, has the Sperner property. E. R. Canfield showed in 1976 [CA] that the answer to Rota's question is "not for n sufficiently large." Follow-on papers by J. B. Shearer [SH], J.C. Sha and D. J. Kleitman [SK] simplified Canfield's argument and lowered the bound on n to 3.4×10^6. None of these studies, however, gave any additional information about the asymptotic growth of Sperner sets (e.g., can they grow significantly faster than the largest rank?).

Meanwhile the present author [HA 1; HA 2; HA 3] developed a notion of morphism for the weighted Sperner problem, showing that Rota's question is equivalent to that for the weighted poset $\Pi_n/S_n = \{\sigma \in Z_+^n \mid \Sigma i\sigma_i = n\}$, partially ordered by the cone $K_n = \{\Sigma\alpha_{i,j}(\delta_i + \delta_j - \delta_{i+j}) \mid \alpha_{i,j} \geq 0\}$, δ_i being 0 except in component i where it is 1, Π_n/S_n has rank function $r(\sigma) = \Sigma_{i=1}^n \sigma_i$, and weights $w(\sigma) = n!/\Pi_{i=1}^n (i!)^{\sigma_i}\sigma_i!$.

Since Π_n/S_n is embedded in R^n, the possibility of passing to a continuous limit presented itself, the limit object being infinite dimensional (a

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function space) partially ordered by a cone and having a Gaussian weight function.

Based upon reasonable hypotheses of convergence, continuity, and form of the solution for the continuous Sperner problem, it was possible to make calculations which gave deep insight into Rota's question. Most striking among these is probably the asymptotic ratio of $1/\sqrt{1-3\sqrt{3}/8} \simeq 1.69$ between the sizes of the largest Sperner set and largest rank of Π_n.

The technical difficulties of a variational problem in infinite dimensional space seemed overwhelming to one accustomed to finite sets, so I set out to study the continuous analog for finite dimensions: The continuous poset Q consists of R^n, partially ordered by a cone K, having rank function $r(x) = x \cdot e$, $e = (1, 1, \dots, 1)$ in the interior of K and Gaussian weight $w(x) = e^{(-1/2)(xC^{-1}x')}/\sqrt{(2\pi)^n \det C}$, where C is an $n \times n$ positive definite matrix.

The problem is to maximize $\omega_N(S) = \int w'(x) g_S(x)\, dx$ over all smooth Sperner sets, S, where $g_S(x) = \min_{y \in K}(y \cdot u / y \cdot e)$, u being the unit normal to S at x with $u \cdot e > 0$. g_S is called "the geometric factor" and enters in because if the surface is not level (i.e., perpendicular to e) at x, then the discrete Sperner set which it approximates must be terraced following the contour of the land given by S. When the terraced set jumps up a rank, some vertices near S must be eliminated in order to maintain incomparability. $g_S(x)$ is the proportion of those that remain.

In [HA3] we showed that for our finite dimensional analog of the continuous Rota problem, $eC \in K$ implies the Sperner property, i.e., that $Q_0 = \{x \in R^n | x \cdot e = 0\}$ is the maximum weight Sperner set in Q. If $eC \notin K$ we conjectured that Q_0 is no longer the solution but that the solutions is still a hyperplane $V_u = \{x \in R^n | x \cdot u = 0\}$, u being a unit vector in $K^* = \{y \in R^n \colon x \cdot y \geq 0\ \forall x \in K\}$, the dual of K. The purpose of this paper is to prove the conjecture.

At first even this finite dimensional restriction of the continuous Rota problem seemed impossibly hard: It is an n-dimensional calculus of variations problem with infinitely many constraints. It is not clear that there need be a smooth solution. However, after the completion of further research several encouraging insights presented themselves:

1. In working out the solution to the continuous analog of a problem by Kleitman and West, I learned that special methods may succeed where general ones do not, if the form of the solution is a simple one (like a hyperplane).

2. The realization that duality in the finite Sperner problem could be carried over to the continuous one.

3. The discovery of papers by Jacobs, which present a continuous analog of the Ford–Fulkerson theory of flows on networks.

2. Flows

The (finite) Sperner problem is known to be dual to the minflow problem in the Hasse diagram of the partial order. This suggests that we should investigate continuous analogs of the minflow problem. Fortunately K. Jacobs has already looked into continuous analogs of the Ford–Fulkerson theory of flows in networks [JS], [JA]. He investigated maxflow problems but this difference is inconsequential. Following Jacobs, but with some variations, we have our basic definitions: A *(continuous) network* N consists of

1. R^n, the coordinatized n-dimensional vector space
2. A cone $K \subseteq R^n$
3. The vector $e = (1, 1, \ldots, 1)$, in the interior of K, giving the direction of flow
4. A continuous weight function w': $R^n \to R^+$.

A *path* π in N is (the graph of) a function π: $R \to R^n$ such that $\pi(s) \cdot e = s$ and for all $s < t, \pi(t) - \pi(s) \in K$. This implies that $|\pi(t) - \pi(s)| < C|t - s|$ for some positive constant C. π is Lipschitz, therefore continuous, differentiable almost everywhere, and rectifiable. Where it exists, the tangent vector τ is in K. Let Π be the set of all paths in N.

A *flow* in N is a measure, f, on Π. Let $P(\pi, \cdot\,)$ denote the rank-difference measure on the graph of π. This means that if $C \subseteq R^n$ is a measurable set such that $\pi \cap C$ is an arc, say the image of the interval from s to t under π, then $P(\pi, C) = \pi(t) \cdot e - \pi(s) \cdot e = t - s$. P is thus a kernel from Π to R^n. It transports measures from Π to R^n and bounded measurable functions from Π to R^n in the usual fashion. Our P differs from that of Jacobs which is defined as the arc-length measure. The density of the rank-difference measure is less than that of Jacobs arc-length measure by a factor of $e \cdot \tau$ at each $x \in \pi \cap C$ (τ being the unit tangent of π at x).

Let $w \geq 0$ be any measure on R^n. Our continuous minflow problem can then be stated: Among all flows, f, in N satisfying $\int_\Pi (P(\pi, \cdot\,)\, df(\pi) \geq w$ find the minimum net flow $f(\Pi)$.

This problem can be interpreted physically as follows: A fluid flows at a unit rate in the direction e. As it moves it can redistribute itself (along the paths of Π) in directions perpendicular to e but the total velocity must always remain in K. Assuming that w has a density w' (is absolutely continuous with respect to Lebesque measure) then the density of the flow (in direction e) at each point, x, must be at least $w'(x)$. What is the minimum net flow with which this can be accomplished?

3. Weak Duality

Our problem is a measure-theoretical linear program. The dual program is: among all bounded, measurable functions h: $R^n \to R^+$, maximize $\int_{R^n} h(x)\, dw(x)$ subject to the condition $\int_{R^n} P(\pi, x)h(x)\, dx \leq 1$ for all $\pi \in \Pi$.

Weak Duality Theorem. *For all flows f and functions h as above, $f(\Pi) \geq \int_{R^n} w'(x)h(x)\, dx$.*

Proof.

$$\begin{aligned} f(\Pi) &= \int_{\Pi} 1\, df(\pi) \\ &\geq \int_{\Pi}\int_{R^n} P(\pi, x)h(x)\, dx\, df(\pi) \\ &= \int_{R^n}\left(\int_{\Pi} P(\pi, x)\, df(\pi)\right)h(x)\, dx \\ &\geq \int_{R^n} w'(x)h(x)\, dx. \end{aligned}$$

Q.E.D.

4. Sperner Sets

A Sperner set is characterized by meeting each path π in at most one point. We shall restrict our attention to smooth Sperner sets, i.e., those which are differentiable $(n-1)$-manifolds. In order for a differentiable manifold to be a Sperner set it is necessary that the unit normal at each point be in $K^* = \{y \in R^n\colon x \cdot y \geq 0\ \forall x \in K\}$, the dual of K. This local condition is not sufficient but it is convenient to extend our problem to include all manifolds satisfying this condition.

Also define $w_N(S) = \int_S w'(x)g_S(x)\, dx$, where integration is with respect to the $(n-1)$-form of S and $g_S(x) = \min_{y \in K}(y \cdot u / y \cdot e)$, u being the unit normal to S at x with $u \cdot e > 0$. Then

Theorem. *Let f be a flow on N and S be a smooth Sperner set. Then $\int_{\Pi} P(\pi, \cdot\,)\, df(\pi) \geq w$ implies $f(\Pi) \geq w_N(S)$.*

We present the proof only for Sperner sets which are hyperplanes $V_u = \{x \in R^n: x \cdot u = 0\}$. The essential idea is the same for curved surfaces, but the analytic details are subtler. Anyway, this special case is the only one needed in the rest of the paper.

Proof. If $S = V_u$, a hyperplane, then $g_S(x) = \min_{y \in K}(y \cdot u/y \cdot e) = g_u$, a constant. Given $\varepsilon > 0$, let $S_\varepsilon = \{x \in R^n: \exists y_1, y_2 \in S,\ x \geq y_1$, and $\|x - y_2\| \leq \varepsilon\}$. Noting that for all

$$\pi \in \Pi, \qquad \int_{S_\varepsilon} P(\pi, x)\, dx \leq \varepsilon \max_{y \in K}(y \cdot e/y \cdot u) = \varepsilon/g_u$$

we have $\int_{R^n} P(\pi, x)(\chi_{S_\varepsilon}(x) g_u/\varepsilon)\, dx = \int_{S_u} P(\pi, x)(g_u/\varepsilon)\, dx \leq 1$, where $\chi_S(x)$ is 1 if $x \in S$ and 0 otherwise. Letting $h = (g_u/\varepsilon)\chi_{S_\varepsilon}$ in the Weak Duality Theorem, we have

$$\begin{aligned} f(\Pi) &\geq \int_{R^n} w'(x) h(x)\, dx \\ &= (1/\varepsilon) \int_{S_\varepsilon} w'(x) g_u\, dx \\ &= \int_{V_u} \left((1/\varepsilon) \int_0^\varepsilon w'(y + su) g_u\, ds \right) dy, \qquad \text{since } S_\varepsilon \simeq [0, \varepsilon] \times V_u \\ &\to \int_{V_u} w'(y) g_u\, dy, \qquad \text{as } \varepsilon \to 0^+ \end{aligned}$$

by the Fundamental Theorem of Calculus, since w' is continuous. Q.E.D.

This inequality supports the claim that our continuous analog of flow is the correct one for studying the continuous Sperner problem. If we find a Sperner set S and flow f such that $f(\Pi) = w_N(S)$, then both S and f must be optimal. The identification would be complete if we could show that such S and f exist for every continuous network N, but consideration of this will be left to another paper.

5. Gaussian Weights

Having established a general framework for our problem we now return to the case which motivated the investigation:

$$w'(x) = e^{-(1/2)(xC^{-1}x')} / \sqrt{(2\pi)^n \det(C)},$$

where C is an $n \times n$ positive definite matrix. If S is a hyperplane, $S = \{x: x \cdot u = 0\}$, u being the unit normal, and $u \in K^*$, then by direct calculation,

$$
\begin{aligned}
w_N(S) &= \int_S w'(x) g_u(x)\, dx \\
&= g_u \int_S w'(x)\, dx \\
&= g_u / \sqrt{2\pi u C u^t}\,,
\end{aligned}
$$

where $g_u = \min_{x \in K}(x \cdot u / x \cdot e)$.

Now there are two subcases to consider:

Case I. $eC \in K$. Let $u = e$, so $S_u = R_0$, the hyperplane of vectors having rank 0. Also let f be a flow on the paths in the fixed direction $d = eC/eCe^t \in K$. Then $d \cdot e = 1$ and for every $x_0 \in R_0$ we have a unique path $\pi(x) = x_0 + sd$, $-\infty < s < \infty$. Define f on these paths by the density $f'(\pi) = w'(x_0)$.

The Jacobian of $y = \frac{1}{2}(xC^{-1}x^t)$ is $\partial y / \partial x = xC^{-1}$. Thus any vector n normal to the surface $\frac{1}{2}(xC^{-1}x^t) = \frac{1}{2}(x_0 C^{-1} x_0^t)$ at $x = x_0$ must satisfy the equation $(x_0 C^{-1}) \cdot n = 0$. Note that

$$
\begin{aligned}
(x_0 C^{-1}) \cdot d &= (x_0 C^{-1}) d^t \\
&= (x_0 C^{-1}) \left(\frac{eC}{eCe^t} \right)^t \\
&= \frac{1}{eCe^t} (x_0 C^{-1} C e^t), \qquad \text{since } C^t = C \\
&= \frac{1}{eCe^t} (x_0 \cdot e) = 0.
\end{aligned}
$$

Therefore w' takes its maximum value on $\pi(s) = x_0 + sd$ at x_0. The density of $\int_\Pi P(\pi, \cdot\,)\, df(\pi)$ takes this same value on all of π, so $\int_\Pi P(\pi, \cdot\,)\, df(\pi) \geq w$. However $f(\Pi) = \int_{Q_0} w'(x)\, dx = w_N(Q_0)$, since $g_e = 1$. Therefore Q_0 is a maximum weight Sperner set.

The argument in Case I gives another proof of the main theorem of [HA3], i.e., that if $eC \in K$ then the continuous weighted poset has the Sperner property. Actually we believe that $eC \in K$ is equivalent to the LYM property, but this must await a definition of the LYM property for continuous posets.

Case II. Suppose that V_{u_0} is the hyperplane maximizing $g_u / \sqrt{2\pi(uCu^t)}$, that $u_0 \neq e$, and that $u_0 \in K^* - \partial K^*$, the interior of

K^*. Then the Jacobian of $(g_u)^2/uCu^t$ exists at u_0 and must be 0, i.e.,

$$0 = \frac{\partial}{\partial u}\left(\frac{(g_u)^2}{uCu^t}\right)\Bigg|_{u=u_0} = \frac{u_0Cu_0^t\left[2g_{u_0}(\partial g/\partial u)(u_0)\right] - (g_{u_0})^2(u_0C)}{\left(u_0Cu_0^t\right)^2}.$$

Thus, we have that

$$\frac{\partial g}{\partial u}(u_0) = \frac{(g_{u_0})^2(u_0C)}{u_0Cu_0^t(2g_{u_0})} = \lambda_0(u_0C),$$

where λ_0 is a constant.

Again we consider a unidirectional flow, this time in the direction d such that $d \cdot e = 1$ and $d \cdot u_0 = g_{u_0}$. Assume that K is strictly convex and smooth, so that d is a differentiable function of u_0 if $u_0 \neq e$. Then $(\partial g/\partial u)(u_0) = (\partial/\partial u)(du^t)|_{u=u_0} = d + (\partial d/\partial u)(u_0)u_0^t$. But the last term represents the way in which d changes as u changes from u_0, the change being in the same direction. Since the solution set of a linear program is not changed by multiplying the objective function by a positive constant, that term is 0, and we have $(\partial g/\partial u)(u_0) = d$ if $u_0 \neq e$.

Coupling the equations of the previous two paragraphs we have, $d = (\partial g/\partial u)(u_0) = \lambda_0(u_0C)$.

Now, suppose $x_0 \in V_{u_0}$, i.e., $x_0 \cdot u_0 = 0$. Then the Jacobian of $H(x) = \frac{1}{2}(xC^{-1}x^t)$ is xC^{-1}, so d is in the tangent space of the ellipsoid $H(x) = H(x_0)$ at $x = x_0$ if and only if $(x_0C^{-1}) \cdot d = 0$. And, in fact,

$$\begin{aligned} x_0C^{-1} \cdot d &= x_0C^{-1}\, d^t \\ &= x_0C^{-1}(\lambda_0 u_0 C)^t \\ &= \lambda_0\left(x_0C^{-1}\right)\left(C^t u_0^t\right) \\ &= \lambda_0 x_0(C^{-1}C)u_0^t, \qquad \text{since } C^t = C \\ &= \lambda_0\left(x_0u_0^t\right) \\ &= \lambda_0(x_0 \cdot u_0) \\ &= 0. \end{aligned}$$

Thus the maximum value that w' takes on each path π in direction d is at the point where it intersects V_{u_0}. As in Case I, we define a density for f on Q_0 by this maximum value and we have $\int_\Pi P(\pi, x)\, df(\pi) \geq w$.

This density may be transformed to one on V_{u_0}, multiplying by $d \cdot u_0 = g_{u_0}$. Thus $\int_{Q_0} \max_s w'(x + sd)\, dx = \int_{V_{u_0}} w'(x)g_{u_0}\, dx = w_N(V_{u_0})$ and so by weak duality V_{u_0} is a maximum weight Sperner set. (*Note.* $G(u) = (g_u)^2/uCu^t$ is not differentiable at e, but it does have directional derivatives.) By the methods of Case II we may show that if $eC \notin K$ then the directional derivative of G at e in the direction $-eC$ is positive. Thus e is not even a local maximum for it, i.e., Q_0 is not a maximum weight Sperner set.

6. Comments and Conclusions

The interaction between combinatorial and continuous mathematics has fascinated me for many years. Several people who worked on this Rota conjecture have characterized the lattice of partitions as being chaotic. The solution of the finite dimensional continuous Rota problem, which is presented in this paper, is the first, that I know of, of a Sperner problem for a family of posets without the Sperner property. And the solution is so simple that it should be extendable to infinite dimensions. All this shows again the power of analysis to bring order out of chaos.

Jacobs and Seiffert [JS] attempted unsuccessfully to prove a continuous analog of the maxflow = mincut theorem of Ford–Fulkerson. However, in formulating the definition of this paper I had the advantage of basing it on a concrete example, the continuous limit of Π_n/S_n. Also, I had the advantage of refering to Jacobs' papers. Consequently, J. Chavez and I have been able to prove a continuous analog of the maxflow = mincut theorem, which we shall report on in another paper.

In order to complete the projected asymptotic solution of the Rota problem, it will be necessary to study convergence and approximation for networks. Jacobs [JA] has shown the way here also, but our definition of flow is different, so it must be recast. Even given the proper theoretical framework, the proof of convergence will surely be a hard calculation.

Variants of the finite dimensional results of this paper would be interesting, especially if applications were found: There are other unimodal distributions in probability theory as well as sums of Gaussian distributions (which would be multimodal) to consider, though I would expect to encounter greater technical difficulties in their solution.

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